MATH 210: Introduction to Analysis

Fall 2015-2016, Midterm 2, Duration: 60 min.

Exercise 1.

(a) (**10 points**) State the definition of a metric space.
(b) (**10 points**) Prove that $(\mathbb{R}^2, d_\infty)$ is a metric space.

Exercise 2.

(1) (**8 points**) Prove that the series $\sum_{n\geq 1} a_n$ with $a_n = \frac{1}{\sqrt{n+1}} - \frac{1}{\sqrt{n}}$ is convergent and compute its sum.
(2) (**12 points**) Show that the series $\sum \frac{a^n}{n}$ converges if and only if $-1 \leq a < 1$.
(3) **5 points.** Assume that the series $\sum a_n^2$ and $\sum b_n^2$ converge. Prove that the series $\sum a_n b_n$ converges absolutely.

Exercise 3.

(a) (**10 points**) Find the interior and the closure of $\mathbb{Z}$. Explain clearly to ensure full credits.
(b) (**10 points**) Show that $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\} \cap \{(x, y) \in \mathbb{R}^2 \mid y \geq x^2\}$ is compact. Explain clearly to ensure full credits.

Exercise 4. We say that a sequence $\{(a_n, b_n)\}$ of $\mathbb{R}^2$ *blows up* if $d_\infty((a_n, b_n), (0, 0))$ diverges to $+\infty$.

(a) (**4 points**) Prove or disprove using an explicit counterexample that if a sequence $\{(a_n, b_n)\}$ blows up then $|a_n|$ and $|b_n|$ diverge to $+\infty$.
(b) (**4 points**) Prove that if a subset $A \subset \mathbb{R}^2$ is unbounded then there is a sequence $\{(a_n, b_n)\}$ of elements of A that blows up.
(c) (**4 points**) Prove that if a subset $A \subset \mathbb{R}^2$ contains a sequence $\{(a_n, b_n)\}$ that blows up then A is unbounded.
(d) Prove that the following sets are unbounded.
   i. (**4 points**) $\mathbb{R} \times (0, 1) = \{(x, y) \in \mathbb{R}^2 \mid 0 < y < 1\}$.
   ii. (**4 points**) $\{(x, y) \in \mathbb{R}^2 \mid x^2 = y^2\}$.

Exercise 5. Let $\{x_n\}$ be a sequence of real numbers.

(a) In this question, we suppose that the series $\sum |x_{n+1} - x_n|$ converges. The goal of this question is to prove that $\{x_n\}$ converges. Denote by S_N the partial sum of $\sum |x_{n+1} - x_n|$ of order N. Let $\varepsilon > 0$.
   i. (**5 points**) Explain very briefly why there an integer N_0 such that for any integers $M, N \geq N_0$ we have $|S_M - S_N| < \varepsilon$.
   ii. (**5 points**) Assume that $M \geq N$. Show that

$$|S_M - S_N| = |x_{N+1} - x_N| + |x_{N+2} - x_{N+1}| + \cdots + |x_{M+1} - x_M|.$$

iii. (**5 points**) Deduce that for $M \geq N \geq N_0$ we have $|x_{M+1} - x_N| < \varepsilon$. (hint: write $x_{M+1} - x_N$ as a telescopic sum).

iv. (**5 points**) Deduce that $\{x_n\}$ converges.

(b) (**5 points**) Give an example of a sequence $\{x_n\}$ such that $|x_{n+1} - x_n|$ converges to zero but $\{x_n\}$ diverges.